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ABSTRACT

The purpose was to ascertain whether there existed differential achievement effects associated with the factors and levels of context, form, item-stem and item-response format, size of common number, and school grade, as they relate to two variations of distributivity. Twelve 9-item tests were constructed and given to intact 4th-, 5th-, 6th-, and 7th-grade classes from two midwestern city school districts; each pupil worked with one of the 12 randomly distributed tests. Conclusions were that pupils' sensitivity to the use of distributivity was relatively low, that ability to complete correctly examples of types tested tended to increase from grade four to five to six to seven, that "regrouping sets" examples were less difficult than "multiplication-addition" examples and that "right-distributive" examples were easier than "left-distributive" examples, and that pupils had only a limited tendency to respond in the same way across a set of examples within a test. (DT)

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SOME FACTORS ASSOCIATED WITH PUPILS' PERFORMANCE ON EXAMPLES INVOLVING SELECTED VARIATIONS OF THE DISTRIBUTIVE IDEA

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Condensation of a report presented at the 1973 Annual Meeting of the American Educational Research Association (New Orleans, 27 February) in a session co-sponsored by Division C and the Special Interest Group for Research in Mathematics Education (SIG/RME).

[In this condensation, findings are summarized in tabular form without elaboration or discussion which will be incorporated in the oral presentation.]

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WHAT ARE THE NUMBERS?

Experimental Copy

		0 8

Name _____ Boy / Girl

School _____ City _____

Grade _____ Math Teacher _____ Class _____

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WORDS	SIGNS	NUMBERS such as	BLANKS
and	x		
are	+	4	
of	=	13	____
sets	( )		

080 0000

Wait

READ CAREFULLY. Complete this example:

$$(6 \times 8) + (6 \times 5) = \_\_\_\_ \times \_\_\_\_$$

Go on to Page 2

Complete this example:

$(8 \times 7) + (8 \times 9) = \_\_\_\_ \times \_\_\_\_$

Go on to Page 3

Complete this example:

$$(9 \times 12) + (5 \times 12) = \_\_\_\_ \times \_\_\_\_$$

Go on to Page 4

Complete this example:

$$(3 \times 14) + (8 \times 14) = \_\_\_\_ \times \_\_\_\_$$

Go on to Page 5

READ CAREFULLY. Complete this example:

5 sets of 9 and 5 sets of 7 are _____ sets of _____

Go on to Page 6

Complete this example:

8 sets of 7

and 4 sets of 7

_____ sets of _____

Go on to Page 7

Complete this example:

15 sets of 5

and 15 sets of 8

_____

____ sets of ____

Go on to Page 8

Complete this example:

6 sets of 16 and 8 sets of 16 are _____ sets of _____

Go on to Page 9

Complete this example:

9 sets of 13

and 9 sets of 13

____ sets of ____

Stop! Close your booklet

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WHAT ARE THE NUMBERS?

		1 1

Name _____ Boy / Girl

School _____ City _____

Grade _____ Math Teacher _____ Class _____

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Center No. C-03/Contract OE 5-10-54

WORDS	SIGNS	NUMBERS such as	BLANKS
and	x		
are	+	4	____
of	=	13	
sets	( )		

110 0000

Wait

READ CAREFULLY. Complete this example:

5 sets of 9

and 5 sets of 8

____ sets of 17

111 1121

Go on to Page 2

Complete this example:

9 sets of 4 and 9 sets of 7 are 9 sets of ____

112 1112

Go on to Page 3

Complete this example:

7 sets of 14 and 5 sets of 14 are 12 sets of ____

113 1212

Go on to Page 4

Complete this example:

	2	sets of	15
and	9	sets of	15
	____	sets of	15

Go on to Page 5

READ CAREFULLY. Complete this example:

$$(5 \times 6) + (8 \times 6) = \_\_\_\_ \times 6$$

Go on to Page 6

Complete this example:

$$(7 \times 9) + (7 \times 5) = \_\_\_\_ \times 14$$

Go on to Page 7

Complete this example:

$$\begin{array}{rrcrr} & (\ 7 & \times & 12\ ) \\ + & (\ 9 & \times & 12\ ) \\ \hline & 16 & \times & \_\_\_\_ \end{array}$$

Go on to Page 8

Complete this example:

$$\begin{array}{rrrr} & (\,16 & \times & 8\,) \\ + & (\,16 & \times & 6\,) \\ \hline & 16 & \times & \underline{\quad\quad} \end{array}$$

Go on to Page 9

Complete this example:

$$(8 \times 13) + (8 \times 13) = \_\_\_\_ \times \_\_\_\_$$

Stop! Close your booklet

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Appended sample Tests. #

* Also see p. 15a for related Table 8a which has been added.
** Also see p. 16a for related Table 9a which has been added.
\# Distributed separately from this document.

Purpose: To ascertain whether there existed differential achievement effects associated with the factors and levels identified in Table as they relate to two variations of distributivity which may be symbolized as

$$(k,p),(k,q) \longrightarrow (k,r) \quad \text{and}$$
$$(p,k),(q,k) \longrightarrow (r,k)$$

where k, p, q, r are counting numbers such that $r = p + q$ and where indicated ordered-pair mappings are translated into words or symbols appropriate to levels of factor A (context), Table 1. [Also see Figure 1.]

Rationale. Sundry variations of distributivity are used in explicit or implicit ways within contemporary elementary-school mathematics programs.[1] However, little if any systematic consideration appears to have been given to the possibility of differential effects associated with some of the Table 1 factors and their potential interactions. This is so both for textbook material that has been prepared for elementary-school pupils and for the limited empirical research that has been reported (e.g., Crawford, 1965; Flournoy, 1964; Gray, 1965; Schell, 1968). It very well may be that some of elementary-school pupils' alleged difficulty with "the distributive idea" is to be found in failure of mathematics programs to take into account conceptual differences inherent in certain of the factors and levels examined in the present investigation.

Procedure. Twelve 9-item tests were constructed in accord with Table 2 to incorporate levels of factors A thru E identified in Table 1. (Copies of one of Tests 01 - 08 and one of Tests 09 - 12 are appended.) Early in the fall of 1972 these tests were given to intact 4th-, 5th-, 6th-, and 7th-grade classes from two midwestern city school districts. In each district all tests were administered by one person (not a regular classroom teacher), and each participating pupil worked with one of the 12 randomly distributed tests under a 12-minute working-time limit. Directions for completing the test items purposely avoided any mention of distributivity or its potential application. Pupils were instructed to progress thru the test booklets page by page (item by item) and were not permitted to turn back at any time to a previously completed or attempted or omitted page (item).

Partial findings. The data reported here are drawn from Tests 01-08 only, and especially from items 1-8 of those tests. (See Table 3 and Table 4.)

A cursory preliminary examination of completed test booklets suggested that it would be desirable to devise a scheme to categorize observed pupil responses.

The scheme adopted may be illustrated in part using "(4,8),(7,8)→ ___,___" to symbolize an item in right-distributive form, with a wholly-open response format, whose context is either regrouping sets or multiplication-addition, and whose item-stem format is either horizontal or vertical; e.g.,

4 sets of 8 and 7 sets of 8 are ___ sets of ___, OR

(4 x 8) + (7 x 8) = ___ x ___, OR

	4	sets of	8	OR		(4	x	8)
and	7	sets of	8		+	(7	x	8)
	___	sets of	___			___	x	___

For any one of these variations, some observed pupil responses and their respective codings would be:

11 , 8 --- Criterion response: $(p + q),(k)$*. . . . . Code 1

8 , 11 --- Commuted criterion response: $(k),(p + q)$ . . . . . Code 1C

2 , 44 --- Some other mathematically correct response, (a,b), where $ab = k(p + q)$ but $a \neq k$, $b \neq k$ . . . . . Code 1A

28 , 64 --- $(pq),(k^2)$ . . . . . Code 2

64 , 28 --- $(k^2),(pq)$ . . . . . Code 2C

32 , 56 --- $(pk),(qk)$ . . . . . Code 3

56 , 32 --- $(qk),(pk)$ . . . . . Code 3C

12 , 15 --- $(p + k),(q + k)$ . . . . . Code 4

15 , 12 --- $(q + k),(p + k)$ . . . . . Code 4C

11 , 16 --- $(p + q),(2k)$ . . . . . Code 5

16 , 11 --- $(2k),(p + q)$ . . . . . Code 5C

If (for the preceding illustrative examples) a pupil response was not among the ones coded above, it was designated either "Code 6" or "Code 9" in accord with conditions identified later in connection with Table 8 and Table 9.

* If a pupil's observed response on a test item is identical with the criterion response for that item, this is no guarantee that distributivity was applied by the pupil in formulating his response. Recognition and use of distributivity is, alas, not a necessary condition for giving a criterion response.

When interpreting the data to be presented, certain intra-district comparisons may be made validly, but inter-district comparisons generally are unwarranted for a variety of reasons.

Table 5 summarizes for Districts 1 and 2 pupil performance in terms of criterion responses. Table 6 (for District 1) and Table 7 (for District 2) summarize performance in terms of the more inclusive mathematically correct responses. The extremely limited number of observed criterion responses as well as other mathematically correct responses, and the related curtailed variance of any distribution of such responses, made it senseless to examine the data in terms of the factorial design originally planned. Instead, attention was directed to a consideration of pupils' apparent misconceptions associated with items of the kind investigated.

Data pertaining to some of these apparent misconceptions are presented in Table 8 (for District 1) and Table 9 (for District 2), without regard for Table 1 factors and levels (other than F, school grade). A more detailed indication of the relative frequency of incorrect responses in relation to item characteristics is found in Table 11 (for District 1) and Table 12 (for District 2). [Information regarding the special case of item 9 is presented in Table 10.]

A. Each of the following null hypotheses was tested independently with the chi-square statistic, where df = (4 - 1)(7 - 1) = 18 and where H_0 was rejected in favor of H_1 (proportions not the same for all grades) at $\alpha = .05$, for which $\chi^2 \geq 28.87$.

H_0: The proportion of observed pupil responses in categories 1, 2, 3, 4, 5, 6, 9 is the same at all grade levels (4, 5, 6, 7) in a given District for ...

1. Item type 1113: Regrouping sets context, Left-distributive form, Horizontal item-stem format, Wholly-open item-response format.
2. Item type 1123: Regrouping sets context, Left-distributive form, Vertical item-stem format, Wholly-open item-response format.
3. Item type 1213: Regrouping sets context, Right-distributive form, Horizontal item-stem format, Wholly-open item-response format.
4. Item type 1223: Regrouping sets context, Right-distributive form, Vertical item-stem format, Wholly-open item-response format.
5. Item type 2113: Multiplication-addition context, Left-distributive form, Horizontal item-stem format, Wholly-open item-response format.
6. Item type 2123: Multiplication-addition context, Left-distributive form, Vertical item-stem format, Wholly-open item-response format.
7. Item type 2213: Multiplication-addition context, Right-distributive form, Horizontal item-stem format, Wholly-open item-response format.
8. Item type 2223: Multiplication-addition context, Right-distributive form, Vertical item-stem format, Wholly-open item-response format.

Results from these null-hypothesis tests are summarized in Table 13.

B. Let P(1113,2113), for instance, be the probability that observed pupil responses for item types 1113 and 2113 fall in the same category.

Each of the following null hypotheses was tested independently for each District at each grade level with the Cochran Q statistic, where df = 4 - 1 = 3, and where H_0 was rejected in favor of H_1 (non-equality of probabilities) at $\alpha = .05$, for which $\chi^2 \geq 7.82$.

$H_0(1)$: P(1113,2113) = P(1213,2213) = P(1123,2123) = P(1223,2223).

$H_0(2)$: P(1113,1213) = P(1123,1223) = P(2113,2213) = P(2123,2223).

$H_0(3)$: P(1113,1123) = P(1213,1223) = P(2113,2123) = P(2213,2223).

Thus, for $H_0(1)$, paired item-types differ only in their characteristics with respect to factor A (Table 1); for $H_0(2)$, paired item-types differ only in their characteristics with respect to factor B; and for $H_0(3)$, paired item-types differ only in their characteristics with respect to factor C.

Results from these null-hypothesis tests are summarized in Table 14.

Further analyses of the data are in progress.

Some tentative conclusions and conjectures

The following observations are believed to be valid for the samples investigated (which admittedly were not drawn randomly from the school populations involved) and generally hold for both District 1 (in which a clearly "modern" basal textbook series was used) and District 2 (in which a more or less "transitional" basal textbook series was used).

1. As measured by the ability to give criterion responses to examples of the kinds tested, pupils' sensitivity to the use of distributivity was relatively low. The design of the investigation did not enable one to infer whether this was due to pupils' simply not "knowing" the distributive idea or to their failure to recognize its application in the situations presented in the Tests.

2. Pupils' ability to complete correctly examples of the types tested tended to increase from grade 4 to 5 to 6 to 7. It is not at all clear to what extent this may be accounted for by "opportunity to learn" (a factor which also would have a bearing on the preceding observation).

3. Across grade levels it appears that "regrouping sets" examples were less difficult than "multiplication-addition" examples, and that "right-distributive" examples were easier than "left-distributive" examples. These differences, however, were due principally to differences associated with right- and left-

distributive forms within the context of regrouping sets (and not within the multiplication-addition context).

4. Based upon the response categorizations used in this investigation, pupils had only a limited tendency to respond in the same way across the set of examples within a Test (i.e., to have 75% or more of a pupil's responses fall within a single category).

5. The nature of pupils' "incorrect" ("conceptually erroneous") responses which fell in categories 2, 3, 4, or 5 seemed to be determined more by the context of the example (regrouping sets vs. multiplication-addition) or by its format (horizontal vs. vertical), and certain combinations of these factor levels, than by distributive form (left vs. right).

6. It is conjectured that the progressive decrease from grade 4 to 7 in the relative frequency of category 6 responses is due in large measure to a progressive decrease in pupils' computational errors, which in turn accounts for the progressive increase in category 3 and category 5 relative response frequencies for certain factor-level combinations. (A separate examination of individual pupil's category 6 responses is being carried out in an attempt to check on this conjecture indirectly.)

Some broader conjectures are advanced:

1. Pupils are more sensitive to a conceptual distinction between m sets of n and n sets of m than they are to a conceptual distinction between $m \times n$ and $n \times m$.

2. The relationship between m sets of n and $m \times n$ is less well established in pupils' thinking than we likely assume it to be.

3. The principal factors and levels of interest in this normative investigation need to be given more conscious recognition and consideration in the preparation of text and related instructional materials, and in the process of instruction itself.

4. Work with properties which give "structure" to some particular aspect of mathematics is no guarantee that pupils will be exempt from rote learning and "symbol pushing."

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Schell, Leo M. Learning the distributive property by third graders. School Science and Mathematics 68: 28-32; January 1968.

Siegel, Sidney. Nonparametric statistics for the behavioral sciences. New York: McGraw-Hill, 1956.

Wiles, Clyde A., Romberg, Thomas A., and Moser, James M. The relative effectiveness of two different instructional sequences designed to teach the addition and subtraction algorithms. Technical Report No. 222. Madison: Wisconsin Research and Development Center for Cognitive Learning, The University of Wisconsin, June 1972.

Footnote

[1] One application of distributivity that is found commonly in school mathematics programs may be illustrated by an example such as

$$\begin{array}{rcl} 23 & = & (2 \times 10) + 3 \\ +\ \underline{56} & = & \underline{(5 \times 10) + 6} \\ & & (7 \times 10) + 9 \ = \ 79, \end{array}$$

where a right-hand distributive form is embedded within a vertical addition algorithm.

For instance, in a recent report by Wiles, Romberg, and Moser (1972) pertaining to the learning of addition and subtraction algorits, one variation of distributivity clearly is included (but not identified by name) in subordinate pupil behaviors such as

"1. Given the numeral phrase $qb + sb$, states the sentence $qb + sb = (q + s)b$. [And]

"2. Given the numerals $qb + r$ and $sb + t$ in expanded notation, states the sentence

$$\begin{array}{rcl} q & b + & r \\ s & b + & t \\ \hline (q + s)b & + & (r + t) \end{array}$$ [p. 23],"

which is the generalization illustrated by the specific example cited at the outset.

In the present investigation interest focused on things akin to 1, however, rather than 2.

TABLE 1

Factors and Levels Investigated

WHAT ARE THE NUMBERS?, Tests 01-12, Items 1-9

Factor	Level
A. Context	1. Regrouping sets
	2. Multiplication-addition
B. Distributive form	1. Left: (k,p), (k,q)
	2. Right: (p,k), (q,k)
	*[3. Ambiguous: (i,j), (i,j)]
C. Item-stem format	1. Horizontal
	2. Vertical
D. Item-response format	**[1. Left-open: __a__ , b]
	**[2. Right-open: a, __b__]
	3. Wholly open: __a__ , __b__
E. Size of common number	1. $4 < k < 10$; $1 < p < 10$, $1 < q < 10$, $p + q > 10$
	2. $11 < k < 17$; $1 < p < 10$, $1 < q < 10$, $p + q > 10$
	*[3. $4 < i < 10$, $11 < j < 17$; or $11 < i < 17$, $4 < j < 10$]
F. School grade	1. Beginning 4th
	2. Beginning 5th
	3. Beginning 6th
	4. Beginning 7th

Notes.-- * Applies to item 9 only, for all Tests.

** Applies to Tests 09-12 only, for items 1 thru 8.

TABLE 2

Characteristics of Test Items Relative to Levels of Factors A, B, C, D, E

Test	Item #1	Item #2	Item #3	Item #4	Item #5	Item #6	Item #7	Item #8	Item #9
	-------Regrouping sets------				-------Multiplication-addition------				
01	RV	LH	LV	RH	RV	RH	LH	LV	AH
03	RH	RV	LH	LV	LH	LV	RV	RH	AV
05	LV	RH	RV	LH	RH	LH	LV	RV	AH
07	LH	LV	RH	RV	LV	RV	RH	LH	AV
09	*RV	*LV	LH*	RH*	*LH	*RH	LV*	RV*	AH
11	LV*	*LH	*RH	RV*	RH*	LH*	*RV	*LV	AV
	---Multiplication-addition---				----------Regrouping sets----------				
02	RH	LV	LH	RV	LV	RH	LH	RV	AH
04	LH	RV	RH	LV	RH	LV	RV	LH	AV
06	RV	RH	LV	LH	RV	LH	RH	LV	AH
08	LV	LH	RV	RH	LH	RV	LV	RH	AV
10	RV*	LV*	*RH	*LH	LH*	RH*	*RV	*LV	AH
12	*LV	*RV	LH*	RH*	*RH	RV*	LV*	*LH	AV

Notes.--

Key for levels of factor B: 1. L = Left-distributive form
2. R = Right-distributive form
3. A = Ambiguous distributive form

Key for levels of factor C: 1. H = Horizontal item-stem format
2. V = Vertical item-stem format

All items of Tests 01 thru 08, and item 9 of Tests 09 thru 12, reflect level 3 of factor D. For items 1 thru 8 of Tests 09 thru 12, an asterisk (*) at the right of a letter-pair indicates level 1 of factor D; an asterisk at the left of a letter-pair indicates level 2 of factor D.

Items 1, 2, 5 and 6 of each Test reflect level 1 of factor E; items 3, 4, 7 and 8 of each Test reflect level 2 of factor E.

TABLE 3

Levels 1 and 2 of Factors A, B, C in Conjunction with Level 3 of Factor D

A. Context	B. Distributive form	C. Item-stem format	
		1. Horizontal	2. Vertical
1. Regrouping sets	1. Left	k sets of p and k sets of q are ___ sets of ___	k sets of p and k sets of q ___ sets of ___
	2. Right	p sets of k and q sets of k are ___ sets of ___	p sets of k and q sets of k ___ sets of ___
2. Multiplication-addition	1. Left	(k × p) + (k × q) = ___ × ___	k × p + k × q ___ × ___
	2. Right	(p × k) + (q × k) = ___ ___	p × k + q × k ___ × ___

Note.--These are the kinds of distributivity variations covered by items 1-8 of Tests 01-08, which also reflect levels 1 and 2 of factor E as defined in Table 1 and referred to in the last Note accompanying Table 2.

TABLE 4

Sources of Data
Tests 01-08

	Number							
	District 1				District 2			
	Grade 4	Grade 5	Grade 6	Grade 7	Grade 4	Grade 5	Grade 6	Grade 7
Participating schools	2	2	1	1	7	7	7	2
Classes tested	7	10	10	9	17 (3[a])	18 (3[b])	19	18
Pupils drawn for data analysis, Tests 01-08	128	176	232	208	288	336	336	320
Pupils per Test[c]	16	22	29	26	36	42	42	40

[a] In each of these three classes, some pupils were fourth graders; others were fifth graders.

[b] In each of these three classes, some pupils were fifth graders; others were sixth graders.

[c] In each instance the number of pupils per Test was partitioned as equally as possible among the schools from which the pupils were drawn.

TABLE 5

Performance Based on Criterion Responses, Tests 01-08

Factor and level	Mean number of criterion responses							
	District 1				District 2			
Items 1-8	Grade 4	Grade 5	Grade 6	Grade 7	Grade 4	Grade 5	Grade 6	Grade 7
A1. Regrouping sets context	.133	.551	.457	.606	.090	.202	.339	.625
A2. Multiplication-addition context	.023	.176	.233	.226	.045	.068	.089	.238
B1. Left-distributive form	.063	.290	.220	.226	.031	.074	.107	.200
B2. Right-distributive form	.094	.438	.470	.606	.104	.196	.321	.663
C1. Horizontal item-stem format	.078	.347	.341	.423	.076	.134	.214	.441
C2. Vertical item-stem format	.078	.381	.349	.409	.059	.137	.214	.422
E1. $4 < k < 10$	.086	.324	.345	.413	.094	.173	.238	.481
E2. $11 < k < 17$	.070	.403	.345	.418	.042	.098	.190	.381
All 8 items	.156	.727	.690	.832	.135	.271	.428	.863
Item 9	.008	.068	.069	.139	.014	.018	.033	.113

Note.--For items 1-8, the maximum mean number of criterion responses possible for each of the 4-item factor levels is 4.000 (and 8.000 for the 8 items collectively); for item 9, the maximum is 1.000.

TABLE 6

Performance Based on Mathematically Correct Responses, District 1

[Items 1-8, Tests 01-08]

Item characteristics			Percent of criterion responses (and other responses coded 1C or 1A)			
Context	Distributive form	Item-stem format	Grade 4	Grade 5	Grade 6	Grade 7
Regrouping sets	Left	Horizontal	3.1 * (3.1)	8.5 (4.5)	4.7 (4.3)	6.2 (6.7)
		Vertical	2.3 (0.0)	10.2 (2.3)	6.9 (1.3)	5.8 (3.4)
	Right	Horizontal	3.9 (0.0)	18.2 (0.6)	16.4 (0.4)	24.0 (1.4)
		Vertical	3.9 (0.8)	18.2 (0.0)	17.7 (0.0)	24.5 (1.4)
Multiplication-addition	Left	Horizontal	0.0 (0.0)	4.5 (1.1)	5.6 (4.3)	4.8 (7.2)
		Vertical	0.8 (0.0)	5.7 (0.0)	4.7 (2.2)	5.8 (3.4)
	Right	Horizontal	0.8 (0.8)	3.4 (1.1)	7.3 (3.4)	7.2 (4.8)
		Vertical	0.8 (0.0)	4.0 (0.6)	5.6 (2.6)	4.8 (2.9)

Note.--*The sum of 3.1 [percent of criterion responses, Code 1] and 3.1 [percent of other correct responses, Codes 1C or 1A] is the percent of mathematically correct responses [6.2]. This condition prevails throughout the Table.

TABLE 7

Performance Based on Mathematically Correct Responses, District 2
[Items 1-8, Tests 01-08]

Item characteristics			Percent of criterion responses (and other responses coded 1C or 1A)			
Context	Distributive form	Item-stem format	Grade 4	Grade 5	Grade 6	Grade 7
Regrouping sets	Left	Horizontal	0.7 * (1.0)	1.8 (6.2)	4.2 (5.1)	4.1 (10.3)
		Vertical	0.7 (0.3)	2.4 (2.7)	3.0 (3.9)	5.9 (6.9)
	Right	Horizontal	3.8 (0.3)	8.0 (0.9)	12.8 (0.9)	27.5 (1.3)
		Vertical	3.8 (0.0)	8.0 (0.9)	14.0 (1.2)	25.0 (2.2)
Multiplication-addition	Left	Horizontal	0.7 (1.0)	2.1 (5.1)	3.5 (5.4)	5.3 (10.0)
		Vertical	1.0 (0.3)	1.2 (3.0)	2.1 (2.1)	4.7 (6.6)
	Right	Horizontal	2.4 (0.3)	1.5 (2.1)	3.0 (3.6)	7.2 (8.1)
		Vertical	0.3 (0.0)	2.1 (2.7)	2.4 (2.4)	6.6 (4.1)

Note.--*The sum of 0.7 [percent of criterion responses, Code 1] and 1.0 [percent of other correct responses, Codes 1C or 1A] is the percent of mathematically correct responses [1.7]. This condition prevails throughout the Table.

TABLE 8

Distribution of Categorized Pupil Responses, District 1
[Items 1-8, Tests 01-08]

Category of pupil response (a,b) relative to item stem (k,p),(k,q) or (p,k),(q,k)	Percent of responses across items and tests			
	Grade 4	Grade 5	Grade 6	Grade 7
1. (a = k), (b = p + q); or (a = p + q), (b = k); or ab = k(p + q), a ≠ k, b ≠ k	2.5	10.4	10.9	14.3
2. (a = k^2),(b = pq); or (a = pq),(b = k^2)	2.1	2.3	3.6	2.6
3. (a = kp),(b = kq); or (a = kq),(b = kp)	12.2	19.3	27.4	30.8
4. (a = k + p),(b = k + q); or (a = k + q),(b = k + p)	5.2	6.3	4.7	4.6
5. (a = 2k),(b = p + q); or (a = p + q),(b = 2k)	18.8	25.4	23.1	24.8
6. (a,b) is none of the above, and (a < 1000),(b < 1000)	41.4	28.6	23.2	18.0
9. (a,b) is none of the above	17.8	7.7	7.2	5.0

Note.--Category 6 is a confounded mixture of responses which are:
(1) conceptually different from any of categories 1 thru 5 and computationally correct; or
(2) conceptually different from any of categories 1 thru 5 and computationally incorrect; or
(3) conceptually the same as one or another of categories 1 thru 5 but computationally incorrect.

Category 9 also is a confounded mixture of responses: all which do no fit into one or another of categories 1 thru 6.

See Table 10 for an extension of this Table as it relates to item 9.

TABLE 8a

Distribution of Pupil Responses [Items 1-8, Tests 01-08] by Factor and Level, District 1

Grade	Factor	Level	Response category						
			1	2	3	4	5	6	9
4	Context	Regrouping sets	4.3	0.6	6.3	8.0	24.0	41.4	15.4
		Multiplication-addition	0.8	3.7	18.2	2.3	13.5	41.4	20.1
	Distributive form	Left	2.3	2.7	11.5	4.5	19.7	40.8	18.4
		Right	2.7	1.6	12.9	5.9	17.8	42.0	17.2
	Item-stem format	Horizontal	2.9	0.0	15.2	8.6	7.4	43.8	22.1
		Vertical	2.1	4.3	9.2	1.8	30.1	39.1	13.5
5	Context	Regrouping sets	15.6	0.4	6.3	11.6	32.2	26.1	7.7
		Multiplication-addition	5.1	4.1	32.4	1.0	18.6	31.1	7.7
	Distributive form	Left	9.2	2.0	19.7	6.7	26.3	27.8	8.2
		Right	11.5	2.6	18.9	6.0	24.6	29.4	7.1
	Item-stem format	Horizontal	10.5	0.0	25.4	10.2	11.5	32.5	9.8
		Vertical	10.2	4.5	13.2	2.4	39.3	24.7	5.5
6	Context	Regrouping sets	12.9	1.5	12.8	8.9	31.4	24.2	8.2
		Multiplication-addition	8.9	5.6	42.1	0.4	14.8	22.1	6.1
	Distributive form	Left	8.5	3.2	28.0	4.4	25.1	23.1	7.7
		Right	13.4	3.9	26.8	5.0	21.0	23.3	6.7
	Item-stem format	Horizontal	11.6	1.0	32.7	7.1	14.0	26.2	7.4
		Vertical	10.2	6.1	22.2	2.3	32.1	20.2	6.9
7	Context	Regrouping sets	18.4	1.3	11.9	8.4	36.1	17.4	6.5
		Multiplication-addition	10.2	4.0	49.6	0.7	13.5	18.5	3.5
	Distributive form	Left	10.8	2.4	31.9	4.9	26.0	18.3	5.8
		Right	17.8	2.9	29.7	4.2	23.6	17.7	4.2
	Item-stem format	Horizontal	15.6	0.7	34.1	6.3	17.3	20.3	5.6
		Vertical	13.0	4.6	27.4	2.9	32.2	15.6	4.3

TABLE 9

Distribution of Categorized Pupil Responses, District 2
[Items 1-8, Tests 01-08]

Category of pupil response (a,b) relative to item stem (k,p),(k,q) or (p,k),(q,k)	Percent of responses across items and tests			
	Grade 4	Grade 5	Grade 6	Grade 7
1. (a = k), (b = p + q); or (a = p + q), (b = k); or ab = k(p + q), a ≠ k, b ≠ k	2.1	6.3	8.4	17.0
2. ($a = k^2$), (b = pq); or (a = pq), ($b = k^2$)	0.8	1.4	2.1	1.9
3. (a = kp), (b = kq); or (a = kq), (b = kp)	10.1	22.8	27.2	24.5
4. (a = k + p), (b = k + q); or (a = k + q), (b = k + p)	4.6	3.7	4.2	4.3
5. (a = 2k), (b = p + q); or (a = p + q), (b = 2k)	14.7	11.8	20.3	26.9
6. (a,b) is none of the above, and (a < 1000), (b < 1000)	38.4	39.3	27.3	18.5
9. (a,b) is none of the above	29.4	14.7	10.5	6.9

Note.--Category 6 is a confounded mixture of responses which are:
(1) conceptually different from any of categories 1 thru 5 and computationally correct; or
(2) conceptually different from any of categories 1 thru 5 and computationally incorrect; or
(3) conceptually the same as one or another of categories 1 thru 5 but computationally incorrect.
Category 9 also is a confounded mixture of responses: all which do not fit into one or another of categories 1 thru 6.

See Table 10 for an extension of this Table as it relates to item 9.

TABLE 9a

Distribution of Pupil Responses [Items 1-8, Tests 01-08] by Factor and Level, District 2

Grade	Factor	Level	Response category						
			1	2	3	4	5	6	9
4	Context	Regrouping sets	2.7	0.2	5.4	6.2	19.3	37.6	28.7
		Multiplication-addition	1.6	1.5	14.8	3.0	10.0	39.1	30.0
	Distributive form	Left	1.5	1.0	10.4	4.7	14.2	37.7	30.6
		Right	2.8	0.7	9.7	4.4	15.0	39.1	28.2
	Item-stem format	Horizontal	2.6	0.0+	11.0	7.0	6.0	41.4	31.9
		Vertical	1.6	1.6	9.1	2.1	23.3	35.3	26.9
5	Context	Regrouping sets	7.7	0.4	14.2	6.3	16.0	40.0	15.4
		Multiplication-addition	4.9	2.4	31.4	1.2	7.6	38.5	13.9
	Distributive form	Left	6.1	1.6	22.8	3.7	11 8	38.8	15.2
		Right	6.5	1.2	22.8	3.7	11.8	39.7	14.1
	Item-stem format	Horizontal	6.9	0.0	25.1	6.1	5.0	41.3	15.6
		Vertical	5.7	2.8	20.5	1.3	18.6	37.3	13.8
6	Context	Regrouping sets	11.2	0.9	15.9	8.0	26.5	26.8	10.7
		Multiplication-addition	5.6	3.3	38.5	0.4	14.1	27.8	10.2
	Distributive form	Left	6.8	2.1	28.1	4.5	21.6	26.7	10.3
		Right	10.0	2.1	26.4	3.9	19.0	27.9	10.6
	Item-stem format	Horizontal	9.1	0.1	32.7	6.6	11.0	28.6	11.9
		Vertical	7.7	4.1	21.7	1.8	29.6	26.0	9.0
7	Context	Regrouping sets	20.8	0.7	9.3	7.1	35.8	18.7	7.7
		Multiplication-addition	13.1	3.0	39.8	1.6	18.0	18.3	6.1
	Distributive form	Left	13.4	2.1	24.7	4.4	28.6	19.1	7.6
		Right	20.5	1.6	24.4	4.3	25.2	17.8	6.2
	Item-stem format	Horizontal	18.4	0.2	29.0	6.9	17.7	20.7	7.1
		Vertical	15.5	3.5	20.0	1.9	36.2	16.3	6.6

TABLE 10

Distribution of Pupil Responses on Item 9, Tests 01-08

Code or category	Percent of responses across Tests							
	District 1				District 2			
	Grade 4	Grade 5	Grade 6	Grade 7	Grade 4	Grade 5	Grade 6	Grade 7
Code 1L[a]	0.8	2.8	2.6	2.4	0.3	0.3	1.2	2.5
Code 1R[b]	0.0	4.0	4.3	11.5	1.0	1.5	2.1	8.8
Code 1LC or 1RC or 1A[c]	0.0	0.0	1.7	5.3	0.3	1.2	1.5	7.2
Category 1	0.8	6.8	8.6	19.2	1.7	3.0	4.8	18.4
Category 2	0.0	0.0	3.4	1.4	0.0	0.3	0.9	0.6
Category 3	10.2	22.7	30.2	29.3	5.9	20.5	25.9	22.2
Category 4	3.1	9.1	6.9	5.3	3.1	3.0	5.1	6.6
Category 5	20.3	24.4	22.8	26.0	11.8	14.9	17.3	25.6
Category 6	43.0	29.0	19.8	13.5	23.3	28.0	25.0	15.0
Category 9	22.7	8.0	8.2	5.3	54.2	30.4	21.1	11.6

Notes.--[a]The criterion response was expressed in left-distributive form.

[b]The criterion response was expressed in right-distributive form.

[c]The mathematically correct response was the commuted version of 1L or 1R, or some other acceptable pair of numbers.

TABLE 11

Distribution of Incorrect Responses, District 1
[Items 1-8, Tests 01-08]

Item characteristics				Percent of responses by category					
Context	Distributive form	Item-stem format	Grade	2	3	4	5	6	9
Regrouping sets	Left	Horizontal	4	0.0	7.0	10.2	7.8	47.7	21.1
			5	0.0	10.2	18.7	14.8	31.8	11.4
			6	0.4	15.9	13.8	23.7	27.2	9.9
			7	0.5	13.5	13.5	28.8	20.7	10.1
		Vertical	4	1.6	3.1	4.7	41.4	35.9	10.9
			5	1.1	2.3	5.7	51.7	22.7	4.0
			6	2.2	9.5	3.4	45.7	21.6	9.5
			7	2.4	11.1	4.3	49.0	17.3	16.7
	Right	Horizontal	4	0.0	7.0	15.6	10.2	46.9	16.4
			5	0.0	6.8	19.3	14.2	30.7	10.2
			6	0.4	13.8	13.8	16.4	31.0	7.8
			7	0.5	10.1	10.6	26.0	21.6	5.8
		Vertical	4	0.8	7.8	1.6	36.7	35.2	13.3
			5	0.6	5.7	2.8	48.3	19.3	5.1
			6	3.0	12.1	4.7	39.7	17.2	5.6
			7	1.9	13.0	5.3	40.4	10.1	3.4
Multiplication-addition	Left	Horizontal	4	0.0	25.8	3.1	7.0	37.5	26.6
			5	0.0	43.2	1.7	9.1	30.1	10.2
			6	1.3	53.0	0.0	8.2	22.0	5.6
			7	1.0	58.2	0.5	6.7	17.8	3.8
		Vertical	4	9.4	10.2	0.0	22.7	42.2	14.8
			5	6.8	23.3	0.6	29.5	26.7	7.4
			6	9.1	33.6	0.4	22.8	21.6	5.6
			7	5.8	44.7	1.4	19.2	17.3	2.4
	Right	Horizontal	4	0.0	21.1	5.5	4.7	43.0	24.2
			5	0.0	41.5	1.1	8.0	37.5	7.4
			6	1.7	47.8	0.9	7.8	24.6	6.5
			7	1.0	54.8	0.5	7.7	21.2	2.9
		Vertical	4	5.5	15.6	0.8	19.5	43.0	14.8
			5	9.7	21.6	0.6	27.8	30.1	5.7
			6	10.3	33.6	0.4	20.3	20.3	6.9
			7	8.2	40.9	0.5	20.2	17.8	4.8

Note.--The "Percent of responses by category" is based upon the total number of observed responses which include those in category 1 (mathematically correct) at each grade level.

TABLE 12

Distribution of Incorrect Responses, District 2
[Items 1-8, Tests 01-08]

Item characteristics				Percent of responses by category					
Context	Distributive form	Item-stem format	Grade	2	3	4	5	6	9
Regrouping sets	Left	Horizontal	4	0.0	6.6	11.1	7.6	38.5	34.4
			5	0.0	15.2	10.1	6.5	43.2	17.0
				0.0	21.4	13.4	14.6	29.5	11.9
				0.0	11.9	12.2	27.2	25.3	9.1
		Vertical	4	0.0	5.2	1.7	29.9	36.1	26.0
			5	0.6	13.7	3.0	24.4	38.7	14.6
			6	1.5	11.9	3.3	42.0	25.3	9.2
			7	2.2	6.9	2.8	51.6	15.6	8.1
	Right	Horizontal	4	0.0	4.5	10.4	7.3	43.7	29.9
			5	0.0	13.7	9.8	6.8	42.9	17.9
			6	0.3	19.0	11.9	13.4	29.2	12.5
			7	0.0	10.9	10.6	21.2	21.2	7.2
		Vertical	4	0.7	5.2	1.4	32.3	31.9	24.7
			5	0.9	14.3	2.1	26.2	35.4	12.2
			6	1.8	11.3	3.3	36.0	23.2	9.2
			7	0.6	7.5	2.8	43.1	12.5	6.3
Multiplication-addition	Left	Horizontal	4	0.3	16.7	3.1	4.5	41.7	31.9
			5	0.0	35.4	1.8	3.3	37.5	14.9
			6	0.0	47.9	0.6	8.0	26.2	10.4
			7	0.0	47.8	1.9	9.7	18.1	7.2
		Vertical	4	3.5	13.2	2.8	14.9	34.4	29.9
			5	5.7	27.1	0.0	12.8	36.0	14.3
			6	6.8	31.0	0.6	21.7	25.9	9.8
			7	6.3	32.2	0.9	25.9	17.5	5.9
	Right	Horizontal	4	0.0	16.3	3.5	4.5	41.7	31.2
			5	0.0	36.0	2.7	3.6	41.7	12.5
			6	0.0	42.6	0.6	8.0	29.5	12.8
			7	0.9	45.3	2.8	12.5	18.1	5.0
		Vertical	4	2.1	12.8	2.4	16.3	38.9	27.1
			5	3.9	27.1	0.3	11.0	39.0	14.0
			6	6.2	32.7	0.0	18.7	29.8	7.7
			7	5.0	33.7	0.9	24.1	19.4	6.3

Note.--The "Percent of responses by category" is based upon the total number of observed responses which include those in category 1 (mathematically correct) at each grade level.

TABLE 13

Summary of Chi-square Tests of Null Hypotheses, Set A

H_0	Item type	District 1		District 2	
		χ^2	Decision about H_0	χ^2	Decision about H_0
1	1113	60.71	Reject	202.18	Reject
2	1123	40.01	Reject	167.81	Reject
3	1213	67.12	Reject	216.16	Reject
4	1223	69.30	Reject	179.10	Reject
5	2113	100.26	Reject	204.47	Reject
6	2123	84.91	Reject	174.21	Reject
7	2213	108.18	Reject	222.37	Reject
8	2223	67.07	Reject	173.23	Reject

Note.--H_0 rejected in favor of H_1 at $\alpha = .05$ when (for df = 18) $\chi^2 \geq 28.87$.

For District 1 the eight values of ϕ' [Cramér's statistic, bounded by 0 (complete independence) and 1 (complete dependence)] ranged from .13 thru .22; For District 2 the eight values of ϕ' ranged from .21 thru .24.

TABLE 14

Summary of Cochran-Q Tests of Null Hypotheses, Set B

H_0	Grade level	District 1		District 2	
		χ^2	Decision about H_0	χ^2	Decision about H_0
1	4	3.17	Not reject	1.59	Not reject
	5	33.60	Reject	2.72	Not reject
	6	1.95	Not reject	3.28	Not reject
	7	6.72	Not reject	16.83	Reject
2	4	0.64	Not reject	3.30	Not reject
	5	12.66	Reject	0.74	Not reject
	6	0.55	Not reject	1.42	Not reject
	7	13.62	Reject	4.60	Not reject
3	4	5.66	Not reject	2.42	Not reject
	5	3.34	Not reject	7.02	Not reject
	6	2.02	Not reject	11.70	Reject
	7	7.70	Not reject	7.15	Not reject

Note.--H_0 rejected in favor of H_1 at $\alpha = .05$ when (for df = 3) $\chi^2 \geq 7.82$.

A simpler gross indication of pupils' tendency to make the same kind of response across items 1 thru 8 of Tests 01 or 02 or . . . or 08 was expressed in terms of the percent of pupils for whom 75% or more of their 8 responses were in a single category (1 or 2 or 3 or 4 or 5 or 6 or 9).

For District 1 these indices of response consistency were 33.6% for grade 4, 17.6% for grade 5, 28.9% for grade 6, and 23.6% for grade 7.

In the case of District 2 these indices were 35.1% for grade 4, 27.4% for grade 5, 27.1% for grade 6, and 24.4% for grade 7.

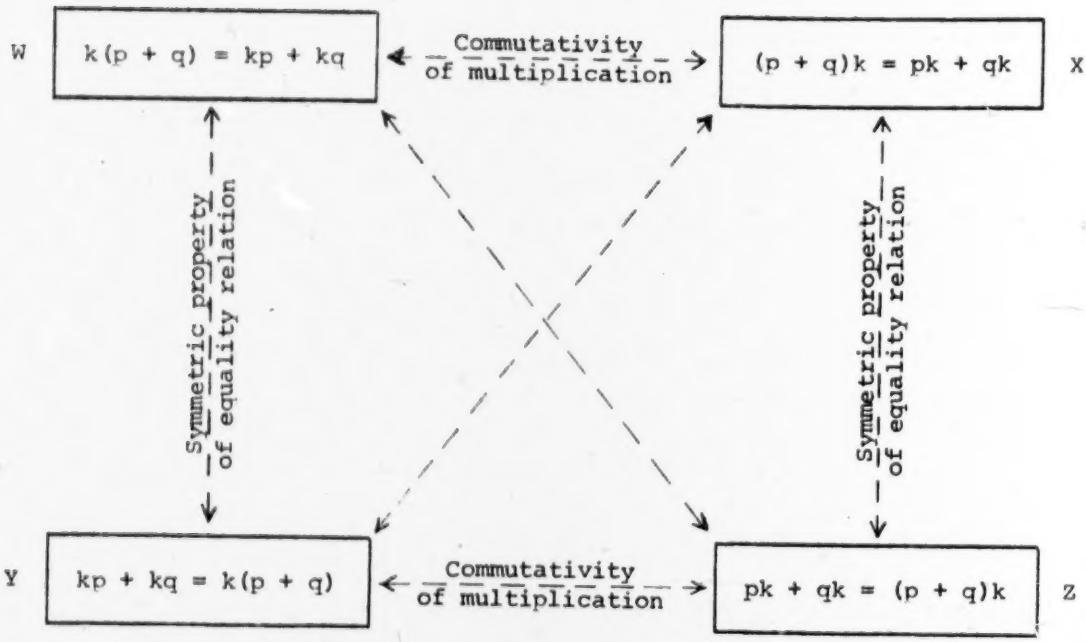
W
k(p + q) = kp + kq
Commutativity of multiplication
(p + q)k = pk + qk
X
Symmetric property of equality relation
Symmetric property of equality relation
Y
kp + kq = k(p + q)
Commutativity of multiplication
pk + qk = (p + q)k
Z

Fig. 1. Four equivalent variations of the distributivity of multiplication over addition for counting numbers k, p, q.

Note.--The variations of distributivity considered in this investigation are directly or analogously associated with Y and Z of Figure 1 rather than with W and X.